

Reply to "Comment" by A. V. Tsiganov

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Abstract

For the Goryachev case we obtain, in the explicit form, the Abel-Jacobi equations with the polynomial of degree six under the radical. We choose the parameters of two families of linear generators of a one sheet hyperboloid to be the separation variables. These variables, as well as the shifted separation variables in the original work of S. Kowalevski, do not commute.

This reply is written as an answer to the paper [1], [2].

1. In [1], [2] Tsiganov states that the variables u_1, u_2 introduced in [3] are not the variables of separation since they do not commute with respect to the initial Poisson bracket even on the zero-level of the area integral. Let us write down all equations obtained in [3], which were not shown by Tsiganov in "Comment".

Denote

$$\begin{aligned} p_1(u) &= 2b + k - u^2, & p_2(u) &= 2b - k + u^2, & p_3(u) &= (u - b)^2 - f^2, \\ p_{ij} &= p_i(u_j), & r_{ij} &= \sqrt{p_{ij}} \quad (i = 1, 2, 3; \quad j = 1, 2). \end{aligned}$$

and suppose that $u_{1,2}$ are the roots of the quadratic equation

$$zu^2 - 2bu + (2b\xi - kz) = 0, \quad z = \alpha_3^2, \quad \xi = M_1^2 + M_2^2 + \frac{b}{\alpha_3^2}. \quad (1)$$

Theorem 2. [3] *The variables u_1, u_2 are separation variables, and their evolution is described by the Abel-Jacobi equations*

$$\frac{du_1}{\sqrt{W(u_1)}} - \frac{du_2}{\sqrt{W(u_2)}} = 0, \quad \frac{u_1 du_1}{\sqrt{W(u_1)}} - \frac{u_2 du_2}{\sqrt{W(u_2)}} = dt, \quad (2)$$

where

$$\begin{aligned} W(u) &= b^{-1} p_1(u) p_2(u) p_3(u) = \\ &= b^{-1} (2b + k - u^2) (2b - k + u^2) [(u - b)^2 - f^2]. \end{aligned}$$

Here the phase variables $\mathbf{M}, \boldsymbol{\alpha}$ are expressed algebraically in terms of u_1, u_2 by the formulas

$$\begin{aligned} M_1 &= i \frac{r_{21} r_{22}}{2\sqrt{2b(u_1 + u_2)}}, & M_2 &= -\frac{r_{11} r_{12}}{2\sqrt{2b(u_1 + u_2)}}, \\ M_3 &= -\frac{i}{2\sqrt{b(u_1^2 - u_2^2)}} (r_{12} r_{22} r_{31} + r_{11} r_{21} r_{32}), \\ \alpha_1 &= \frac{1}{2\sqrt{b(u_1^2 - u_2^2)}} (r_{12} r_{21} r_{31} + r_{11} r_{22} r_{32}), \\ \alpha_2 &= -\frac{i}{2\sqrt{b(u_1^2 - u_2^2)}} (r_{11} r_{22} r_{31} + r_{12} r_{21} r_{32}), \\ \alpha_3 &= \frac{\sqrt{2b}}{\sqrt{u_1 + u_2}}. \end{aligned}$$

This theorem is true. It can be proved by direct substitution without using any other theory. Equations (2) could be obviously written down as Kowalevski-type equations:

$$(u_1 - u_2) \frac{du_1}{dt} = \sqrt{W(u_1)}, \quad (u_1 - u_2) \frac{du_2}{dt} = \sqrt{W(u_2)}.$$

For experts, it is not difficult to conclude whether the variables in these equations are *separation variables* or *not*. In [3] neither Lie-Poisson brackets nor Hamiltonian formalism are mentioned. The only notion of differential equation theory we use is the notion of a first integral.

2. In [1], [2] Tsiganov points out the following: "At $b = 0$ this system and the corresponding variables of separation have been investigated by Chaplygin [4]". Further he writes: "In [7] we proved that the Chaplygin variables remain variables of separation for the Goryachev case at $b \neq 0$ ".

To clarify the situation, let us write down some formulas. The papers [7] and [6] are the same, and we will refer to [7] for definiteness.

For $b = 0$ the Chaplygin separation variables have the form [4]:

$$s_{1,2} = \frac{M_1^2 + M_2^2 \pm h}{c\alpha_3^2}, \quad (3)$$

where $h^2 = (M_1^2 - M_2^2 + c\alpha_3^2)^2 + 4M_1^2 M_2^2$.

The Chaplygin separation variables depend on the value of h , the square of which is the function of dynamic variables and, *at the same time*, (for $b = 0$) the value of the first integral [4].

In [7] the separation variables $q_{1,2}$ are introduced as the roots of the quadratic equation [7, formula (3.8)]:

$$\lambda^2 - \left(\frac{M_1^2 + M_2^2}{\alpha_3^2} + c \right) \lambda + \frac{cM_2^2}{\alpha_3^2} = 0, \quad (4)$$

where c corresponds to the parameter c_2 in the formula (3.8) in [7].

Functional relation between variables (3) and (4) could be easily obtained as follows:

$$q_k = \frac{c}{2}s_k + \frac{c}{2}, \quad k = 1, 2. \quad (5)$$

Section 3.1 in [7] is devoted to separation of variables. At the same time, nothing is said about the direct relation (5) between $q_{1,2}$ and the Chaplygin variables. We emphasize that simple formula (5) is first written here and can not be found in [7]. To the contrary, at the end of Section 3.1 we read: "Remark 2. At $c_4 = 0$, we have reproduced the Chaplygin result...". This means that the relation to Chaplygin's result in [7] is pointed out only for $c_4 = 0$, which corresponds to $b = 0$ in equation (1), and, therefore, the separation variables $q_{1,2}$ are presented by Tsiganov in [7] as new variables of separation. Thus, the statement that the separation of variables in Goryachev problem in terms of Chaplygin variables is proved in [7], *does not represent the fact*. Moreover, in [7] there are no references even to the original paper [5] devoted to this problem. In particular, it is shown in [8], [9], [3] that the integral presented in [7] is not new and can be expressed in terms of Goryachev integral. The history of this question can be found in [10].

The question, whether the variables (3) are the separation variables for the Goryachev case, is the question of definition of separation variables. It could be eventually reduced to the question whether the function

$$h^2 = (M_1^2 - M_2^2 + c\alpha_3^2)^2 + 4M_1^2 M_2^2$$

is the first integral for the Goryachev case ($b \neq 0$)? The answer is evident.

3. In [1], [2] Tsiganov states that "an application of the geometric Kharlamov method to the Goryachev system yields noncommutative "new variables of separation" instead of the standard canonical variables of separation", and, in addition, he writes: "It is a remarkable well-known shift of auxiliary variables $u_{1,2}$, which Kowalevski used in [11] in order to get canonical variables of separation $s_{1,2}$ in her case".

Let us turn to the original paper [11] by Kowalevski, the letter [12] of Kowalevski to Mittag-Leffler, as well as to the original papers [13] and [14] by Kötter and Appelrot.

The system of the first integrals is as follows:

$$\begin{aligned} 2(p^2 + q^2) + r^2 &= 2\gamma_1 + 6l_1, \\ 2(p\gamma_1 + q\gamma_2) + r\gamma_3 &= 2l, \\ \gamma_1^2 + \gamma_2^2 + \gamma_3^2 &= 1, \\ \{(p + iq)^2 + \gamma_1 + i\gamma_2\}\{(p - iq)^2 + \gamma_1 - i\gamma_2\} &= k^2, \end{aligned}$$

where l_1, l and k are real constants of the first integrals.

S. Kowalevski introduced the polynomials:

$$\begin{aligned} R(x_1) &= -x_1^4 + 6l_1x_1^2 + 4lx_1 + 1 - k^2, \\ R(x_2) &= -x_2^4 + 6l_1x_2^2 + 4lx_2 + 1 - k^2, \\ R(x_1, x_2) &= -x_1^2x_2^2 + 6l_1x_1x_2 + 2l(x_1 + x_2) + 1 - k^2. \end{aligned}$$

Here

$$x_1 = p + iq, \quad x_2 = p - iq.$$

In the letter to Mittag-Leffler, the founder of the journal "Acta Mathematica", S. Kowalevski introduced the variables $\frac{1}{2}w_1, \frac{1}{2}w_2$, where

$$\begin{aligned} w_1 &= \frac{R(x_1, x_2) - \sqrt{R(x_1)}\sqrt{R(x_2)}}{(x_1 - x_2)^2}, \\ w_2 &= \frac{R(x_1, x_2) + \sqrt{R(x_1)}\sqrt{R(x_2)}}{(x_1 - x_2)^2}, \end{aligned} \tag{6}$$

which at that moment were not shifted. In terms of these variables the Abel-Jacobi equations are written down [12, p. 166].

In [14, p. 69] Appelrot writes: "With this, Kowalevski made her investigation approximately in the way I show below, though in some moments I make known deviations from her following the example of F. Kötter...". And, further, Appelrot [14, p. 70] points out the following:

"These values w , more precisely, the values s_1 and s_2 that are equal to

$$s_1 = w_1 + 3l_1 \quad \text{and} \quad s_2 = w_2 + 3l_1 \tag{7}$$

are treated as *new variables in the analysis, however, Kowalevski, following closer Weierstrass, denoted by s exactly what I denote by \bar{s}* " (italics and the equations number are mine; in Appelrot's book these equations have the number (10) in Page 70). In fact, the variables (7) were introduced by Kötter [13].

Thus, the variables introduced by S. Kowalevski [11, formulas (9), p. 188], are denoted in [14] by \bar{s} :

$$\begin{aligned}\bar{s}_1 &= \frac{R(x_1, x_2) - \sqrt{R(x_1)}\sqrt{R(x_2)}}{2(x_1 - x_2)^2} + \frac{1}{2}l_1, \\ \bar{s}_2 &= \frac{R(x_1, x_2) + \sqrt{R(x_1)}\sqrt{R(x_2)}}{2(x_1 - x_2)^2} + \frac{1}{2}l_1.\end{aligned}\tag{8}$$

The relation between Kowalevski variables (8) and the variables (6) and (7) can be also found in [14, p. 72]:

$$2\bar{s} + 2l_1 = s = w + 3l_1.$$

It is well known [15], [16] that the variables w_1, w_2 do not commute

$$\{w_1, w_2\} \neq 0,\tag{9}$$

but the variables (7) do commute. They are used in classic works by Kötter [13], Appelrot [14], Zhukovsky [17], Golubev [18], Ipatov [19], and also in modern works devoted to constructing «action-angle» variables (the latter should be canonic by definition, therefore the commutation property is motivated in these studies). Thus,

$$\{s_1, s_2\} \equiv 0.$$

Then

$$\{\bar{s}_1, \bar{s}_2\} = \frac{1}{4}[\{w_1, w_2\} - \{L_1, w_1 - w_2\}]\tag{10}$$

and

$$0 = \{s_1, s_2\} = \{w_1, w_2\} - 3\{L_1, w_1 - w_2\}.\tag{11}$$

Here $L_1 = \frac{1}{3}H$ is the first integral with the constant l_1 (H is the Hamiltonian function or the energy integral). From (11) we derive

$$\{L_1, w_1 - w_2\} = \frac{1}{3}\{w_1, w_2\}.\tag{12}$$

Substituting (12) into the expression (10) and using property (9), which is given in [15], [16] as self-evident, we obtain

$$\{\bar{s}_1, \bar{s}_2\} = \frac{1}{6}\{w_1, w_2\} \neq 0.$$

Thus, in the original paper [11] both pairs of the introduced variables $\frac{1}{2}w_1, \frac{1}{2}w_2$ and \bar{s}_1, \bar{s}_2 (the latter pair is shifted from $\frac{1}{2}w_1, \frac{1}{2}w_2$ by the value $\frac{1}{2}l_1$) *do not commute*. There are no other separation variables in the original Kowalevski paper. The fact that the Kowalevski separation variables do not commute is also mentioned in [20, p. 187].

4. Conclusion.

- In [3] the Abel-Jacobi equations with the polynomial of degree six under the radical are obtained for the Goryachev case in the explicit form. For the separation variables we choose the parameters of two families of linear generators of a one sheet hyperboloid. Whether they should commute or not depends on definitions. As can be seen from the original paper [11], the Kowalevski separation variables (8) *do not commute*.
- In [7] and [6], the fact that the Chaplygin variables in the Goryachev case are separation variables (in any sense) is not mentioned clearly.
- In the papers by Tsiganov cited above there are no any references to the original paper by Goryachev [5] devoted to this problem and to the variants of constructing separation variables given by Borisov and Mamaev [20], [21].

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